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Tits-type alternative for automorphism groups

Ivan Arzhantsev

In 1972, Jacques Tits [3] has proved that any subgroup of the general linear group $\mathrm{GL}(n)$ over a field of characteristic zero either contains a nonabelian free subgroup or is a finite extension of a solvable group. In the first part of the talk we survey known facts on the Tits alternative for groups of regular and birational automorphisms of algebraic varieties. Attempts to prove that the Tits alternative holds for automorphism groups have served as an important incentive to study the structure of subgroups. In the last decades this direction became an active research area with many deep results and intriguing open problems.

The second part of the talk is based on joint works with Mikhail Zaidenberg. Let X be an affine algebraic variety defined over an algebraically closed field of characteristic zero and $\mathbb{G}_a$ be the additive group of the ground filed. Consider a subgroup H of the automorphism group $\mathrm{Aut}(X)$ generated by a finite collection of $\mathbb{G}_a$-subgroups $U_1, \ldots, U_k$. We conjecture that either H contains a nonabelian free subgroup or H is a unipotent affine algebraic group. This conjecture is proved in [1] under assumptions that X is an affine toric variety and the subgroups $U_1, \ldots, U_k$ are normalized by the acting torus. In [2], we show that if X is a complex affine surface then either H contains a nonabelian free subgroup or H is a metabelian unipotent affine algebraic group.

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Rost's invariant g_3 determines Albert algebras arising from the first Tits construction uniquely

Vladimir Chernousov

Abstract. Joint work with A. Lourdeaux and A. Pianzola. We provide new evidence that Serre's question whether cohomological invariants determine Albert algebras/groups of type F_4 uniquely up to isomorphism might have a positive answer.

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Hesselink strata for the spinors of the 15-dimensional space

Alexander Elashvili

Аннотация. In the talk, I will report on recent investigation of Hesselink strata for the 128-dimensional spin representation of the orthogonal group $SO_{15}(\mathbb{C})$.

Introduction

Recently in a joint work with M. Jibladze I have been studying Hesselink stratification of the null-cone of the irreducible 128-dimensional spin representation of the simple complex algebraic group of type B_7.

Our calculations were based on the algorithm described by V. L. Popov in 2003 [5]. The algorithm has been recently implemented by W. De Graaf in the computer algebra system GAP. (The implementation is not yet publicly available, but Prof. De Graaf kindly shared it with us).

1. Previous work

Classification of all orbits of the representations spin_n for $n \leq 12$ has been carried out by J.-I. Igusa [1].

Orbits of spin_{13} have been computed by Kac and Vinberg [2].

Orbits of spin_{14} were described by V. L. Popov [3].

Orbits of spin_{16} have been calculated by Antonyan and Elashvili [4].

2. Short description

According to the output of the program of W. De Graaf, the null-cone of the representation spin_{15} possesses 169 strata. In each of these strata we found a representative of an orbit of maximal possible dimension.

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Twisted conjugacy in Chevalley groups

Shripad M. Garge

Let G be a group and $\phi : G \to G$ be an automorphism. We say that $g, h \in G$ are ϕ-conjugate if $g = ah\phi(a)^{-1}$ for some $a \in G$. It is of interest to know if, for a given (infinite) group G, the number of equivalence classes under ϕ-conjugacy is infinity for every automorphism ϕ of G. We discuss this for various Chevalley groups. A part of this talk is based on the recent work with Dr Mitra ([1]).

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Double cosets NgN of normalizers of maximal tori of simple algebraic groups and orbits of partial actions of Cremona subgroups

Nikolai Gordeev

Abstract. Let G be a simple algebraic group over an algebraically closed field K and let $N = N_G(T)$ be the normalizer of a fixed maximal torus $T \leq G$. Further, let U be the unipotent radical of a fixed Borel subgroup B that contains T and let U^- be the unipotent radical of the opposite Borel subgroup B^-. The Bruhat decomposition implies the decomposition $G = NU^-UN$. The Zariski closed subset $U^-U \subset G$ is isomorphic to the affine space A_K^m where $m = \dim G - \dim T$ is the number of roots in the corresponding root system. Here we construct a subgroup $\mathcal{N} \leq \mathrm{Cr}_m(K)$ that "acts partially" on $A_K^m \approx \mathcal{U}$ and we show that there is one-to-one correspondence between the orbits of such a partial action and the set of double cosets $\{NgN\}$. Here we also calculate the set $\{g_\alpha\}_{\alpha \in \mathfrak{A}} \subset \mathcal{U}$ in the simplest case $G = \mathrm{SL}_2(\mathbb{C})$.

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Width of a congruence subgroup over an arithmetic ring

Pavel Gvozdevsky

Abstract. We give an estimate for the width of the congruence subgroup $\mathrm{SL}(n, \mathcal{O}_S, I)$ in Tits–Vaserstein generators, where $\mathcal{O}_S$ is a localisation of the ring of integers in a number field K. We assume that either K has a real embedding, or the ideal I is prime to the number of roots of unity in K.

Extended abstract

Given a group G with generating set X, the width of G in generators from X is a minimal number N such that any element of G is a product of at most N elements from X. When the width is finite, we say that G admits bounded generation with respect to X.

For special linear group, or more generally for Chevalley groups bounded generation with respect to the elementary generators is known for certain classes of rings. For example this holds for Dedekind domains of arithmetic type, see [1] for the arithmetic case and [3] for the functional case. Moreover, the proofs are effective , i.e they give an estimate for the corresponding width.

In [4], it was proven that for an ideal $I \trianglelefteq \mathcal{O}_S$ the principal congruence subgroup $G(\Phi, \mathcal{O}_S, I)$ of a classical Chevalley group of type Φ with $\operatorname{rk}\Phi \leq 2$ over a Dedekind domain $\mathcal{O}_S$ of arithmetic type has finite width in Tits–Vaserstein generators (i.e. generators of type $x_\alpha(\xi)^{x_{-\alpha}(\zeta)}$, where $\alpha \in \Phi$, $\xi \in I$ and $\zeta \in \mathcal{O}_S$), provided the fraction field of $\mathcal{O}_S$ has a real embedding. However, the proof relies on results from [5], which on its turn are not constructive and do not allow to obtain an explicit estimate of the width in question.

The talk is base on the paper [2] by author, where an effective version of the result from [4] is proved for special linear group, i.e. the width of $\mathrm{SL}(n, \mathcal{O}_S, I)$ in Tits–Vaserstein generators is estimated explicitly.

Namely let $\mathcal{O}$ be the ring of integers in an algebraic number field K. Let D be the discriminant of K and $\mathrm{Cl}(K)$ be its class group. Let m be the number of roots of unity in K. For any rational prime p set $e_p = \mathrm{ord}_p(m)$, i.e. $m = \prod_{\{p\colon e_p>0\}} p^{e_p}$. Further for any rational prime p we denote by L_p the extension of K obtained by adjoining a primitive p^{e_p+1}-th root of unity. Now set

$$\mathbb{S}_{\mathrm{bad}} = \{p \in \mathbb{P}\colon p \mid D \text{ and } \gcd([L_p\colon K], |\operatorname{Cl}(K)|) > 1\},$$

where $\mathbb{P}$ denotes the set of rational primes. Finally, set

$$\Delta = \max_{\delta_1+\delta_2+\delta_3=|\mathbb{S}_{\mathrm{bad}}|} \left(\sum_{i=1}^{3} \max(1, [\ln(\delta_i + 1)/\ln 2]) \right),$$

where maximum is taken over all triples of nonnegative integers δ_1,δ_2,δ_3 with $\delta_1 + \delta_2 + \delta_3 = |\mathbb{S}_{\mathrm{bad}}|$. The main result that will be discussed in the talk is the following theorem.

Theorem 1. *In the notation above, let* S *be a multiplicative system in* $\mathcal{O}$*, let* $\mathcal{O}_S = \mathcal{O}[S^{-1}]$*, and* I *be a non zero ideal in* $\mathcal{O}_S$*. Suppose that either* K *has a real embedding, or* I *is prime to* m*. Let* $n \geq 3$*. Then the width of* $\mathrm{SL}(n, \mathcal{O}_S, I)$ *in Tits–Vaserstein generators* $\{t_{i,j}(\xi)^{t_{j,i}(\zeta)}\colon \xi \in I,\ \zeta \in \mathcal{O}_S\}$ *is at most* $3n(n-1)/2 + 2n + 1632\Delta + 185$*.*

In the talk we discuss the main ideas of the proof and certain corollaries. We also discuss how the noneffective version of the theorem above (i.e. the proof of bounded generation without the explicit bound) can be easily obtain directly from the result of [1].

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Sandpile Graphs and graph algebras

Roozbeh Hazrat

Abstract. We give a down to earth introduction to seemingly two very different topics, one about sandpile models (a model about spreading objects along networks) and the other is how to associate interesting algebras to graphs. We then relate these two topics, via the concept of monoids.

Roozbeh Hazrat

Groups and the Bateman–Horn Conjecture

Gareth A. Jones and Alexander K. Zvonkin

Abstract. A number of open problems, ranging from the twin primes conjecture to the classification of permutation groups of prime degree, depend on whether certain finite sets of polynomials in $\mathbb{Z}[t]$ can simultaneously take prime values for infinitely many $t \in \mathbb{N}$. The Bateman–Horn Conjecture (BHC) provides an estimate for the number of such $t \leq x$ for large $x \in \mathbb{R}$, and although it is unproved (except for the case of a single linear polynomial) these estimates agree closely with experimental evidence. We have applied the BHC to a number of problems in group theory (as well as other areas) to give strong evidence that various families of groups, such as permutation groups $\mathrm{PSL}_n(q)$ of prime degree $(q^n - 1)/(q - 1)$, are infinite.

1. Permutation groups of prime degree

Building on earlier work by Galois and Burnside, the classification of finite simple groups implies a classification of the transitive permutation groups of prime degree (a problem dating back to Lagrange). These groups are:

(a) subgroups of $\mathrm{AGL}_1(p)$ containing the translation group, for primes p;
(b) alternating and symmetric groups A_p and S_p, for primes $p \geq 5$;
(c) $\mathrm{PSL}_2(11)$ and Mathieu groups M_{11} and M_{23}, of degrees $11, 11$ and 23;
(d) subgroups G of $\mathrm{P\Gamma L}_n(q)$ containing $\mathrm{PSL}_n(q)$, in cases when the natural degree $d = (q^n - 1)/(q - 1)$ is prime.

It is unknown whether the degree d in case (d) is prime for infinitely many pairs (n, q). We met this problem in trying to extend the work of Klein on equations of degree 7 and 11 to all primes [5]. Such *projective primes*, as we call them, include the Fermat primes, of the form $q+1 = 2^{2^f}+1$, for $n = 2$ and the Mersenne primes, of the form $2^n - 1$, for $q = 2$. In investigating this problem (see [6] for details), to avoid these very difficult cases we restricted our attention to parameters $n, q \geq 3$.

If we write $q = p^e$, we are asking whether p and

$$d := p^{(n-1)e} + p^{(n-2)e} + \cdots + p^e + 1. \tag{1}$$

can both be prime for infinitely many p. Clearly, this requires n to be prime.

2. Prime values of polynomials

These problems are part of a more general number-theoretic problem concerning prime values of polynomials. In 1857 Bunyakovsky [2] conjectured that a polynomial $f(t) \in \mathbb{Z}[t]$ takes prime values for infinitely many $t \in \mathbb{N}$ if and only if it satisfies three obviously necessary conditions:

1. it has a positive leading coefficien,
2. it is irreducible, and
3. it is not identically zero modulo any prime.

For example, these conditions are satisfied by the polynomial t^2+1, the subject of the Euler–Landau Conjecture on primes of this form. The Bunyakovsky Conjecture has been proved only in the case $\deg f = 1$: this is Dirichlet's Theorem on primes in an arithmetic progression. Schinzel's Hypothesis [7] asserts that polynomials $f_1(t), \dots, f_k(t) \in \mathbb{Z}[t]$ simultaneously take prime values for infinitely many $t \in \mathbb{N}$ if and only if they satisfy the first two Bunyakovsky conditions and their product satisfies the third; again, this is proved only for a single linear polynomial.

In 1962 Bateman and Horn [1], extending earlier work by Hardy and Littlewood [4] on twin primes and related problems, conjectured that the number $Q(x)$ of $t \le x$ such that each $f_i(t)$ is prime is given asymptotically by the estimate

$$Q(x) \sim E(x) := C \int_a^x \frac{dt}{\prod_{i=}^{k} \ln f_i(t)} \quad \text{as} \quad x \to +\infty \tag{2}$$

where

$$C = C(f_1, \dots, f_k) := \prod_{r \text{ prime}} \left(1 - \frac{1}{r}\right)^{-k} \left(1 - \frac{\omega_f(r)}{r}\right), \tag{3}$$

$\omega_f(r)$ is the number of roots of $f := f_1 \dots f_k \bmod (r)$, and a is chosen to avoid singularities of the integral, where some $f_i(t) = 1$. There are good heuristic arguments for these formulae, but no proof (again, apart from the case of the quantified version of Dirichlet's Theorem). If Schinzel's conditions are satisfied, the infinite product in (3) converges to a limit $C > 0$. Since the definite integral in (2) diverges to $+\infty$ as $x \to +\infty$, it follows that $E(x) \to +\infty$ and hence, if the BHC is true, $Q(x) \to +\infty$, proving that there are infinitely many $t \in \mathbb{N}$ with each $f_i(t)$ prime.

As simple examples, taking $f_i(t) = t$, $t+2$ or t, $2t+1$ gives the twin primes and Sophie Germain primes problems. Taking

$$f_i(t) = t, \quad t^{(n-1)e} + t^{(n-2)e} + \dots + t^e + 1 \tag{4}$$

for fixed n and e gives our projective primes problem.

3. Application to permutation groups

We concentrated on the simplest and most frequently arising case of the problem, where $n = 3$ and $e = 1$, so that

$$f_i(t) = t, \quad t^2 + t + 1. \tag{5}$$

Maple evaluates the definite integral in (2) almost instantly. The infinite product in (3) converges slowly, but taking the product over the first 10^9 primes r gives a good approximation to the limit. Table 1 compares the resulting estimates $E(x)$ with the actual numbers $Q(x)$ found by applying the Rabin–Miller primality test to the values $f_i(t)$ (this test is probabilistic, but the chances of an error are negligible).

x	$Q(x)$	$E(x)$	$E(x)/Q(x)$
10^{10}	15 801 827	1.579642126×10^7	0.9996579044
$2 \cdot 10^{10}$	29 684 763	2.968054227×10^7	0.9998578150
$3 \cdot 10^{10}$	42 963 858	4.296235691×10^7	0.9999650617
$4 \cdot 10^{10}$	55 877 571	5.587447496×10^7	0.9999445924
$5 \cdot 10^{10}$	68 522 804	6.852175590×10^7	0.9999847043
$6 \cdot 10^{10}$	80 962 422	8.096382889×10^7	1.0000173771
$7 \cdot 10^{10}$	93 236 613	9.323905289×10^7	1.0000261688
$8 \cdot 10^{10}$	105 372 725	1.053741048×10^8	1.0000130940
$9 \cdot 10^{10}$	117 383 505	1.173885689×10^8	1.0000431394
10^{11}	129 294 308	1.292974079×10^8	1.0000239757

TABLE 1. BHC estimates $E(x)$ and actual numbers $Q(x)$ of primes $p^2 + p + 1$ for primes $p \leq x$.

The accuracy of the estimates is comparable to that in other applications of the BHC, such as to twin or Sophie Germain primes. The evidence for other pairs (n, e), such as $(5, 1)$, is good but less convincing, simply because the primes involved increase so rapidly that only a much smaller number of them are within our computing range. Based on this evidence, we make the following conjecture:

Conjecture 3.1. *For each prime $n \geq 3$ there are infinitely many prime powers q such that $(q^n - 1)/(q - 1)$ is prime .*

Of course this would imply that for each prime $n \geq 3$ there are infinitely many permutation groups $\mathrm{PSL}_n(q)$ of prime degree $(q^n - 1)/(q - 1)$.

4. Other applications to group theory

Similar problems and results arise in connection with extensions of the work in the 1890s of Burnside, Frobenius and Hölder on orders of simple groups, and the classification by Dixon and Zalesskii [3] of primitive linear groups of prime degree.

5. Conclusion

These results strongly suggest that various families of groups, appearing in major classification theorems, are all infinite. There are similar results based on the BHC, in our work and in that of others, in areas ranging from combinatorics to elliptic curves, and from error-correcting codes to fast integer multiplication. Their significance is not just that they demonstrate the existence of many examples (often millions) of various constructions and phenomena, but that the accuracy of the estimates obtained provides strong evidence for the validity of the BHC, and hence for the infinitude of these families. Such results also add further emphasis to the desirability of a proof of the BHC, a prospect which currently seems remote.

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Chow ring of $B\,\mathrm{SO}(2n)$ in characteristic 2

Nikita A. Karpenko

Abstract. For $n \geq 1$, let $\mathrm{SO}(2n)$ be the special orthogonal group given by the standard split nondegenerate $2n$-dimensional quadratic form over a field. The Chow ring $\mathrm{CH}(B\,\mathrm{SO}(2n))$ of its classifying space has been computed for the field of complex numbers in 2000 by R. Field. Arbitrary fields of characteristic $\neq 2$ have been treated, using a different method, in 2006 – by L. A. Molina Rojas and A. Vistoli. Using specialization from characteristic 0, we extend their computation to characteristic 2.

Let G be an affine algebraic group over a field F. The Chow ring $\mathrm{CH}(BG)$ of the classifying space of G, considered systematically for the first time in [8], is the ring of characteristic classes for G, where a characteristic class is a functorial assignment for any G-torsor over a smooth F-variety X of an element in the Chow ring $\mathrm{CH}(X)$ of X.

Example 1. Let G be the general linear group $\mathrm{GL}(d)$ for some $d \geq 1$. A G-torsor over a smooth variety X yields a rank d vector bundle E over X. For $i = 1, \ldots, d$, its ith Chern class $c_i(E)$ is an elements of $\mathrm{CH}^i(X)$ defining a characteristic classes $c_i \in \mathrm{CH}^i(BG)$. By [8], $c_1, \ldots, c_d$ are independent generators of the ring $\mathrm{CH}(BG)$ identifying it with the polynomial ring $\mathbb{Z}[c_1, \ldots, c_d]$.

For arbitrary G, given a faithful representation $G \hookrightarrow \mathrm{GL}(d)$, the pull-back ring homomorphism $\mathrm{CH}(B\,\mathrm{GL}(d)) \to \mathrm{CH}(BG)$ transfers the Chern classes

$$c_1, \ldots, c_d \in \mathrm{CH}(B\,\mathrm{GL}(d))$$

to $\mathrm{CH}(BG)$. Besides, evaluating the characteristic classes for G on the base of the G-torsor

$$\mathrm{GL}(d) \to \mathrm{GL}(d)/G,$$

we get a ring homomorphism $\mathrm{CH}(BG) \to \mathrm{CH}(\mathrm{GL}(d)/G)$.

Theorem 2 ([9, Theorem 5.1])**.** *For any $d \geq 1$ and any faithful G-representation $G \hookrightarrow \mathrm{GL}(d)$, the homomorphism $\mathrm{CH}(BG) \to \mathrm{CH}(\mathrm{GL}(d)/G)$ is surjective; its kernel is the ideal generated by $c_1, \ldots, c_d$.*

Remark 3. Theorem 2 is useful in both directions. First of all, it describes the Chow ring of the quotient variety $\mathrm{GL}(d)/G$ in terms of $\mathrm{CH}(BG)$. On the other hand, any given generators of the ring $\mathrm{CH}(\mathrm{GL}(d)/G)$ can be lifted to $\mathrm{CH}(BG)$; any such lifts together with the Chern classes $c_1, \dots, c_d$ generate the Chow ring $\mathrm{CH}(BG)$.

Example 4. For the orthogonal group $\mathrm{O}(d)$, given by the standard split nondegenerate d-dimensional quadratic form over a field (of arbitrary characteristic), and its standard representation $\mathrm{O}(d) \hookrightarrow \mathrm{GL}(d)$, the quotient $\mathrm{GL}(d)/\mathrm{O}(d)$ is an open subset in an affine space. It follows that $\mathrm{CH}(\mathrm{GL}(d)/\mathrm{O}(d)) = \mathbb{Z}$ and so the ring $\mathrm{CH}(B\mathrm{O}(d))$ is generated by $c_1, \dots, c_d$. In characteristic $\neq 2$, the relations are: $2c_i = 0$ for every odd i, [8, §15]. In characteristic 2, the relations are: $c_i = 0$ for every odd i, [5, Appendix B].

Now let us consider the special orthogonal group $\mathrm{SO}(d)$. For odd d, since $\mathrm{O}(d) = \mu_2 \times \mathrm{SO}(d)$, the ring homomorphism $\mathrm{CH}(B\mathrm{O}(d)) \to \mathrm{CH}(B\,\mathrm{SO}(d))$, induced by the embedding $\mathrm{SO}(d) \hookrightarrow \mathrm{O}(d)$, is surjective. Its kernel is generated by c_1. (In characteristic 2, since $c_1 = 0$, the kernel is trivial.)

The case of even $d = 2n$ is much more difficult. The group $\mathrm{SO}(4)$ – the first nontrivial case – was done over the complex numbers in [7]. The group $\mathrm{SO}(2n)$ for arbitrary n – still over the complex numbers – has been treated in [2] (see also [3]). Over an arbitrary field of characteristic $\neq 2$, the ("same") answer was obtained (by a different method) in [6]. Besides of the Chern classes, the answer involves certain characteristic class $y \in \mathrm{CH}^n(B\,\mathrm{SO}(2n))$ constructed by Edidin and Graham:

Theorem 5 ([6])**.** *For $n \geq 1$, the group $\mathrm{SO}(2n)$, considered over a field of characteristic $\neq 2$, has the Chow ring $\mathrm{CH}(B\,\mathrm{SO}(2n))$ generated by the Chern classes $c_2, c_3, \dots, c_{2n}$ together with the Edidin-Graham characteristic class y. The generators are subject to the following relations:*

$$y^2 = (-1)^n 2^{2n-2} c_{2n} \quad \text{and} \quad 2c_i = 0 = c_i \cdot y \text{ for every odd } i.$$

We prove the analogue of Theorem 5 for characteristic 2. Any given field F of characteristic 2 is the residue field of some characteristic 0 discrete valuation field K, [1, Proposition 5 of §2.3 and Proposition 1 of §2.1 in Chapter IX]. We write $\mathrm{SO}(2n)_K$ and $\mathrm{SO}(2n)_F$ for the special orthogonal group over the respective fields and consider the specialization ring homomorphism

$$\mathrm{CH}(B\,\mathrm{SO}(2n)_K) \to \mathrm{CH}(B\,\mathrm{SO}(2n)_F). \tag{6}$$

To explain the definition of (6), note that by [8, Theorem 1.3], the ring $\mathrm{CH}(BG)$ for an affine algebraic group G over any field is approximated by algebraic varieties over the field. By [8, Remark 1.4] (see also [5, Example 4.1]), in the case of $G = \mathrm{SO}(2n)$ it is enough to consider varieties obtained by base change from smooth schemes over the integers. For such varieties, the specialization homomorphism is a ring homomorphism defined in [4, Example 20.3.1].

Theorem 7. *The specialization homomorphism* (6) *is surjective; its kernel is generated by the odd Chern classes $c_3, c_5, \dots, c_{2n-1}$.*

Theorems 5 and 7 together yield

Corollary 8. *For the special orthogonal group* $\mathrm{SO}(2n)$*, where* $n \geq 1$*, considered over a field of characteristic* 2*, the Chow ring* $\mathrm{CH}(B\,\mathrm{SO}(2n))$ *is generated by the* even *Chern classes* $c_2, c_4, \ldots, c_{2n}$ *together with the specialization* $y \in \mathrm{CH}^n(B\,\mathrm{SO}(2n))$ *of the Edidin-Graham characteristic class. These generators are subject to the unique relation*

$$y^2 = (-1)^n 2^{2n-2} c_{2n}.$$

Surjectivity of the specialization homomorphism is the most subtle part of Theorem 7. By Theorem 2, since the Chern classes specialize to "themselves", it is equivalent to the surjectivity of specialization for the quotient variety $\tilde{X} := \mathrm{GL}(2n)/\,\mathrm{SO}(2n)$. Note that a posteriori, the latter specialization homomorphism turns out to be an isomorphism.

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On the structure of nets over quadratic fields

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Abstract. The structure of nets over quadratic fields is studied.

Introduction

Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic field, $\mathfrak{D}$ be a ring of integers of the quadratic field K, $\sigma = (\sigma_{ij})$ be an irreducible D-net of order $n \geq 2$ over the quadratic field K, where σ_{ij} are $\mathfrak{D}$-modules. We prove that up to conjugation diagonal matrix, all σ_{ij} are fractional ideals of a fixed intermediate subring P, $\mathfrak{D} \subseteq P \subseteq K$, and all diagonal rings coincide with P: $\sigma_{11} = \sigma_{22} = \ldots = \sigma_{nn} = P$, where $\sigma_{ij} \subseteq P$ are integer ideals of a ring P for any $i < j$, if $i > j$, then $P \subseteq \sigma_{ij}$. For any i, j we have $\sigma_{1j} \subseteq \sigma_{ij}$. In [1] it has been proven that for some classes of quadratic fields $\mathbb{Q}(\sqrt{d})$, or more precisely, for numbers d equal

$$-1, \quad -2, \quad -3, \quad -7, \quad -11, \quad -19, \quad 2, \quad 3, \quad 5, \quad 6, \quad 7,$$

$$11, \quad 13, \quad 17, \quad 19, \quad 21, \quad 29, \quad 33, \quad 37, \quad 41, \quad 57, \quad 73,$$

with accuracy up to conjugation diagonal matrix from $D(n, K)$ all σ_{ij} are ideals fixed intermediate subring P, $\mathfrak{D} \subseteq P \subseteq K$. In the above cases, the integer ring $\mathfrak{D}$ of the quadratic field K is the principal ideal domain (see [2, ch.III, §2]), and therefore in the description of nets it is possible to use the results of the work [3], in which a complete description of nets and elementary nets over a field of fractions of a principal ideal domain. Ring of integers $\mathfrak{D}$ arbitrary quadratic field $K = \mathbb{Q}(\sqrt{d})$ is not always the principal ideal domain [2].

Conclusion

In the general case, the ring $\mathfrak{D}$ of the quadratic field $K = \mathbb{Q}(\sqrt{d})$ is the Dedekind domain [4]. This determines the relevance of the proposed study.

A set of additive subgroups $\sigma = (\sigma_{ij})$, $1 \leq i, j \leq n$, of a field K is called a net [5] of order n over K if $\sigma_{ir}\sigma_{rj} \subseteq \sigma_{ij}$ for all values of the index i, r, j. A net

$\sigma = (\sigma_{ij})$ is called irreducible if all additive subgroups σ_{ij} are different from zero. A net $\sigma = (\sigma_{ij})$ is called a D-net if $1 \in \tau_{ii}$, $1 \leq i \leq n$.

Theorem. Let $\sigma = (\sigma_{ij})$ be an irreducible D-net of order $n \geq 2$ over K, where σ_{ij} are $\mathfrak{D}$-modules. Then , for some intermediate subring P, $\mathfrak{D} \subseteq P \subseteq K$, the net $\sigma = (\sigma_{ij})$ is conjugated by a diagonal matrix from $D(n, K)$ with a D-net $\pi = (\pi_{ij})$ of fractional ideals π_{ij} of the ring P and has the form:

$$\pi = \begin{pmatrix} P & \pi_{12} & \pi_{13} & \dots & \pi_{1n} \\ \pi_{21} & P & \pi_{23} & \dots & \pi_{2n} \\ \pi_{31} & \pi_{32} & P & \dots & \pi_{2n} \\ . & . & . & \dots & . \\ \pi_{n1} & \pi_{n2} & \pi_{n3} & \dots & P \end{pmatrix},$$

where $\pi_{ij} \subseteq P$ are the integer ideals of the ring P for any $i < j$, if $i > j$, then $P \subseteq \pi_{ij}$. Further, inclusions $\pi_{1j} \subseteq \pi_{ij}$ take place for any i, j.

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On the structure of some unitary Nil K_1-groups

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Abstract. We introduce several Nil-subgroups of the unitary Bass' nilpotent K_1-group of a unitary ring and study structure of these Nil-groups. Obtaining properties of the Nil-groups are unitary analogues well-known properties of the Bass' nilpotent K_1-group of a ring in algebraic $K-$theory.

Introduction

In the paper, we follow the standard setting and notation of unitary (algebraic) $K-$theory [1, 2]. First we recall basic definitions and results used in the sequel.

Let (R, λ, Λ) be a unitary ring, alias Bak's form ring, where R is an associative ring with 1, equipped with an involution $x \to \bar{x}$. Further, let λ be a central element of R such that $\lambda \cdot \bar{\lambda} = 1$, and let Λ be an additive subgroup of R such that $\{x - \lambda\bar{x}, x \in R\} \leq \Lambda \leq \{x \in R : x = -\lambda\bar{x}\}$. We note that $(R, \bar{\lambda}, \overline{\Lambda})$, where $\overline{\Lambda} = \{\bar{x}, x \in \Lambda\}$, is a unitary ring also. Let us extend the involution to the matrix ring $M_r(R)$ by setting $(a_{ij})^* = (\overline{a_{ji}})$.

Definition 1. A matrix $a(\in M_r(R))$ is said to be $\Lambda-$hermitian if the a is a $(-\lambda)-$hermitian, i.e., $a = -\lambda a^*$, and all diagonal entries of the a belong to Λ. It is obvious, a matrix a is $\Lambda-$hermitian if and only if a^* is $\bar{\Lambda}-$hermitian.

In the paper we write matrices in a block form. Namely, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2r}(R)$ means that the components $a, b, c, d \in M_r(R)$. For a natural number r, we set $I_r^\lambda = \begin{pmatrix} 0 & e_r \\ \lambda e_r & 0 \end{pmatrix}$, where e_r is the identity matrix of degree r. We note, that I_r^λ is an invertible $\Lambda-$Hermitian matrix, and $(I_r^\lambda)^{-1} = (I_r^\lambda)^*$.

Definition 2. A matrix $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2r}(R)$ is said to be unitary if $\alpha^* I_r^\lambda \alpha = I_r^\lambda$; and the α to be $\Lambda-$unitary if moreover the diagonal entries of the matrices a^*c and b^*d are contained in Λ.

The set $U_{2r}^\lambda(R, \Lambda)$ of all $\Lambda-$unitary matrices of degree $2r$ forms a group; it is called the (hyperbolic) $\Lambda-$unitary group.

Denote by $EU^{\lambda}_{2r}(R,\Lambda)$ the subgroup of $U^{\lambda}_{2r}(R,\Lambda)$, generated by the matrices $H(a)=\begin{pmatrix} a & 0 \\ 0 & (a^*)^{-1}\end{pmatrix}=diag(a,(a^*)^{-1})$ (hyperbolic matrix), $\begin{pmatrix} e_r & b \\ 0 & e_r\end{pmatrix}$, $\begin{pmatrix} e_r & 0 \\ c & e_r\end{pmatrix}$, where $a\in E_r(R)$, b is $\overline{\Lambda}-$hermitian, and c is $\Lambda-$hermitian. The group $EU^{\lambda}_{2r}(R,\Lambda)$ is called the elementary (hyperbolic) $\Lambda-$unitary group.

Let us define an embedding $U^{\lambda}_{2r}(R,\Lambda)\to U^{\lambda}_{2r+2}(R,\Lambda)$:

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}\to\begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1\end{pmatrix},$$

and set $U^{\lambda}(R,\Lambda)=\cup U^{\lambda}_{2r}(R,\Lambda)$, and $EU^{\lambda}(R,\Lambda)=\cup EU^{\lambda}_{2r}(R,\Lambda)$.

In view of the unitary analog of the Whitehead lemma ([2], Chap.2, Proposition 3.7), the group $EU^{\lambda}(R,\Lambda)$ coincides with the commutator subgroup of the group $U^{\lambda}(R,\Lambda)$. In particular, the (abelian) group $K_1U^{\lambda}(R,\Lambda)=U^{\lambda}(R,\Lambda)/EU^{\lambda}(R,\Lambda)$ is well defined. The class of a matrix $\alpha\in U^{\lambda}(R,\Lambda)$ in the group $K_1U^{\lambda}(R,\Lambda)$ is denoted by $[\alpha]$. As a result, we obtain a unitary K_1-functor K_1U acting from the category of unitary rings to the category of abelian groups.

We extend the involution to the polynomial ring $R[X]$ by setting $\bar{X}=X$. Then $(R[X],\Lambda[X],\lambda)$ is a unitary ring also.

Definition 3. Denote by $NK_1U^{\lambda}(R,\Lambda)$ the kernel of the (splitting) group epimorphism $K_1U^{\lambda}(R[X],\Lambda[X])\to K_1U^{\lambda}(R,\Lambda)$ induced by unitary surjection of unitary rings $(R[X],\Lambda[X])\to(R,\Lambda):X\to 0$. We say that it is the unitary Bass' nilpotent K_1-group of the unitary ring R.

The following statement from [4] should be considered as a unitary analog of the Higman linearization trick, while, in contrast to the linear case of algebraic $K-$theory, only the upper half of the $\Lambda[X]-$unitary matrix, representing an element of the group $NK_1U^{\lambda}(R,\Lambda)$, linearized. In fact, one can carry out a similar linearization of any of the halves of the representing matrix.

Proposition 1 ([4], Theorem 1). Every element of the group $NK_1U^{\lambda}(R,\Lambda)$ has a representative of the form

$$\begin{pmatrix} e_r-aX & bX \\ -cX^n & e_r+a^*X+\cdots+(a^*)^nX^n\end{pmatrix}(\in U^{\lambda}_{2r}(R[X],\Lambda[X]))$$

for some positive integers r and n, where $a,b,c(\in M_r(R))$ satisfy the following conditions:

1) the matrices b and ab are $\bar{\Lambda}-$hermitian and also $ab=ba^*$;

2) the matrices c and ca are $\Lambda-$hermitian and also $ca=a^*c$;

3) $bc=a^{n+1}$ and $cb=(a^*)^{n+1}$.

It is not difficult to show, that the matrix in Proposition 1 is $\Lambda[X]-$unitary if and only if the conditions 1)-3) are satisfied ([4]).

It is well known ([3], Chap.12, Corollary 5.3), that for an arbitrary associative ring R with 1, every element of the Bass' nilpotent K_1-group $NK_1(R)$ has a unipotent representative of the form $e_r - aX$ for some positive integer r, where $a(\in M_r(R))$ is a nilpotent matrix. A similar result for the unitary Bass' nilpotent group $NK_1U^{\lambda}(R,\Lambda)$ does not generally hold. More exactly, the following statement is fairly.

Proposition 2 ([4], Theorem 2). Let matrices $a, b, c \in M_r(R)$ satisfy the conditions 1)-3) of Proposition 1 for some positive integers r and n. Then, in the notations of Proposition 1, the following statements hold:

1) for $n = 1$, matrix

$$\begin{pmatrix} e_r - aX & bX \\ -cX & e_r + a^*X \end{pmatrix} = e_{2r} - \begin{pmatrix} a & -b \\ c & -a^* \end{pmatrix} X (\in U^{\lambda}_{2r}(R[X], \Lambda[X]))$$

is a unipotent and the matrix $\begin{pmatrix} a & -b \\ c & -a^* \end{pmatrix}$ is a nilpotent matrix of nilpotency degree 2;

2) for $n \geq 2$, matrix

$$\begin{pmatrix} e_r - aX & bX \\ -cX^n & e_r + a^*X + \cdots + (a^*)^n X^n \end{pmatrix} (\in U^{\lambda}_{2r}(R[X], \Lambda[X]))$$

is a unipotent if and only if the matrix $e_r - aX$ is unipotent; in this case, the class of this matrix in the group $NK_1U^{\lambda}(R,\Lambda)$ coincides with the class of the hyperbolic matrix $H(e_r - aX)$.

Now we formulate the main result of the paper.

Theorem. Let α be a nonzero matrix in $M_{2r}(R)$. If $e_{2r} - \alpha X^m \in U^{\lambda}_{2r}(R,\Lambda)$ for some natural number m then α is a nilpotent matrix of nilpotency degree 2; in this case, the matrix α has the form $\begin{pmatrix} a & -b \\ c & -a^* \end{pmatrix}$, where matrices $a, b, c (\in M_r(R))$ satisfy the conditions 1)-3) of Proposition 1.

We introduce several Nil-groups and represent some properties of these Nil-groups, which are unitary analogues well-known properties of the Bass' nilpotent K_1-group of a ring in algebraic $K-$theory [5].

We denote by $UnipK_1U^{\lambda}(R,\Lambda)$ the subgroup of $NK_1U^{\lambda}(R,\Lambda)$ is generated by all elements of the following two types:

1) $[e_{2r} - \alpha X^m]$ for some natural numbers r, m, where $\alpha(\in M_{2r}(R))$ is a nilpotent matrix of nilpotency degree 2;

2) $[H(e_r - aX)]$ for some natural number r, where $a(\in M_r(R))$ is a nilpotent matrix.

Moreover we denote by $Unip_1K_1U^{\lambda}(R,\Lambda)$ (respectively, $Unip_2K_1U^{\lambda}(R,\Lambda)$) the subgroup of $UnipK_1U^{\lambda}(R,\Lambda)$ is generated by all elements of the first type (respectively, second type).

Corollary 1. Let n be a positive integer such that $n = n \cdot 1 = 0$ in the ring R, where 1 is the identity element of R. Then the group $Unip_1K_1U^{\lambda}(R,\Lambda)$ is of a $n-$torsion group.

Corollary 2. If p is a prime number such that $p^k = 0$ in R for some natural number k, then $UnipK_1U^{\lambda}(R,\Lambda)$ is a $p-$group.

Corollary 3. If n is an invertible element of R, then the group $Unip_1K_1U^\lambda(R,\Lambda)$ is a uniquely divisible by n.

In unitary (algebraic) $K-$theory for any unitary ring R there exist two standard group homomorphisms: the hyperbolic homomorphism $H : K_1(R) \longrightarrow K_1U^\lambda(R,\Lambda) : [a] \longrightarrow [H(a)]$ and the forgetful homomorphism $F : K_1U^\lambda(R,\Lambda) \longrightarrow K_1(R) : \alpha mod EU^\lambda(R,\Lambda) \longrightarrow \alpha mod E(R)$.

Corollary 4. Under the condition of the Corollary 3, if moreover either the hyperbolic homomorphism H or the forgetful homomorphism F is a monomorphism, then the group $Unip_2K_1U^\lambda(R,\Lambda)$ is a uniquely divisible by n.

Corollary 5. If R is a $\mathbf{Q}-$algebra, where $\mathbf{Q}$ denotes the field of rational numbers, then $Unip_1K_1U^\lambda(R,\Lambda)$ is a $\mathbf{Q}-$vector space. Moreover if either the homomorphism H or the homomorphism F is a monomorphism, then the group $Unip_2K_1U^\lambda(R,\Lambda)$ is a $\mathbf{Q}-$vector space also.

Corollary 6. Under the condition of the Corollary 5, the groups $Unip_1K_1U^\lambda(R,\Lambda)$ and $Unip_2K_1U^\lambda(R,\Lambda))$ are a divisible groups. In particular, these groups are a direct summand both the group $NK_1U^\lambda(R,\Lambda)$ and the group $UnipK_1U^\lambda(R,\Lambda)$.

Conclusion

In Theorem, the main result of the paper, a system of unitary unipotent matrices is found. Using the system of matrices obtained in Theorem, we introduce several Nil-subgroups of the unitary Bass' nilpotent K_1-group of a unitary ring and represent some properties of these Nil-groups. These properties are unitary analogues well-known properties of the Bass' nilpotent K_1-group of a ring in algebraic $K-$theory [5].

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Non-alternating Hamiltonian Lie algebras

A.V. Kondrateva and M.I Kuznetsov

Abstract. The general theory of non-alternating Hamiltonian Lie algebras over a perfect field of characteristic two is discussed. The complex of symmetric differential forms over an algebra of divided powers is constructed. The classification of graded algebras is given and their filtered deformations are found. The classes of equivalence of non-alternating Hamiltonian forms with polynomial coefficients in divided powers are described.

Introduction

Well known Lie algebras of Cartan type consisting of vector fields preserving special, Hamiltonian or contact form have the analogous in characteristic p. Note that the algebra of functions must be replaced by an algebra of divided powers $\mathcal{O}_n(\mathcal{F})$ (see [1]) that corresponds to some generalized flag $\mathcal{F}$ of the space E, $\mathcal{F} : E = E_0 \supseteq E_1 \supseteq \ldots$. Let $\{x_1, \ldots, x_n\}$ be a basis of E coordinated with $\mathcal{F}$ then $\mathcal{O}_n(\mathcal{F}) = \mathcal{O}(n : \mathbf{m})$ where $\mathbf{m} = (m_1, \ldots, m_n)$ is the n-tuple of heights of variables $x_1, \ldots, x_n$ with respect to the flag $\mathcal{F}$. In the case of a field of characteristic 2 one can construct a large class of simple Hamiltonian Lie algebras corresponding to non-alternating symmetric differential forms.

The class of non-alternating Hamiltonian Lie algebras may present interest for, at least, two reasons. First, it can be attributed to algebras of Cartan type when $p = 2$, which is of great importance for the classification of simple Lie algebras of low characteristics. Second, these Lie algebras may admit non-singular derivations, thus may be of interest for p-group theory.

Non-alternating Hamiltonian algebras over a field of characteristic 2 were first constructed in 1993 by Lin Lei [2] as Lie algebras of polynomials in divided powers with the symmetric Poisson bracket $\{f, g\} = \sum\limits_i \partial_i f \partial_i g$. In the case when the heights of the variables are equal to 1, non-alternating Hamiltonian Lie algebras are isomorphic to the first series of simple Lie algebras built by I. Kaplansky [3]. In [4, 5], symmetric differential forms in divided powers are introduced and non-alternating Hamiltonian Lie algebras similar to Hamiltonian Lie superalgebras of characteristic zero with respect to the standard Poisson brackets are studied. Note that the classification of alternating Hamiltonian forms over an algebra of truncated polynomials is obtained by M.I. Kuznetsov, S.A. Kirillov ([6]). The complete classification over a divided powers algebra is built by S.M. Skryabin ([7, 8]).

In the talk, the general theory of non-alternating Hamiltonian Lie algebras in divided powers over a perfect field of characteristic 2 is discussed. The main results may be found in [9]. In [10] the invariant construction of the complex of symmetric differential forms in characteristic 2 was given and some program of investigation was proposed. In the first stage the authors have obtained all invariants of symmetric Hamiltonian differential forms with constant coefficients with respect to parabolic subgroup of $GL(V)$ corresponding to flag $\mathcal{F}$. In particular, it was shown that there exists a basis of V coordinated with flag $\mathcal{F}$ such that a form has a matrix $diag(M_0, \ldots, M_0,\ M_1, \ldots, M_1,\ 1_s)$ where

$$M_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } M_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Later on, employing the theory of truncated coinduced modules ([11, 12]) the authors have proved that in the case when $n > 4$ or $n = 4$ and the flag $\mathcal{F}$ is non-trivial or $n = 2,\ 3$ and E_1 contains a non-isotropic vector with respect to the form $\overline{\omega}$ on E dual to the form ω, each filtered Lie algebra associated with a graded non-alternating Hamiltonian algebra is given by a non-alternating Hamiltonian differential form with non-constant polynomial coefficients in divided powers.

One of the main results is the proof of the equivalence of the non-alternating Hamiltonian form ω with polynomial coefficients to its initial form $\omega(0)$, provided that the canonical form of $\omega(0)$ contains $(dx_i)^{(2)}$ or $dx_i dx_j + (dx_j)^{(2)}$ for some variable x_i of the height greater than 1. The authors present some results on the problem of the classification of non-alternating Hamiltonian forms for which this condition is not fulfilled. In this case the non-alternating Hamiltonian form ω may not be equivalent to $\omega(0)$.

Theorem. *Let ω be a non-alternating Hamiltonian form over the algebra of divided powers $\mathcal{O}_n(\mathcal{F})$, E^0 be the subspace of all isotropic vectors of E with respect to form $\overline{\omega(0)}$ on E dual to $\omega(0)$.*
(i) if $E_1 \not\subset E^0$ then ω is equivalent to $\omega(0)$;
(ii) if $E_1 \subset E^0$ then there exists a basis of E coordinated with $\mathcal{F}$ such that ω is equivalent to

$$dx_1 dx_2 + \ldots + dx_{n-2} dx_{n-1} + dx_n^{(2)} + \sum_{j=1}^{n-1} b_{jn} dz_j dz_n \quad \textit{if } n{=}2k{+}1$$

and

$$dx_1 dx_2 + \ldots + dx_{n-1} dx_n + dx_n^{(2)} + \sum_{j \neq n-1} b_{j,n-1} dz_j dz_{n-1} \quad \textit{if } n{=}2k$$

Moreover, if $b_{jn} \neq 0$ *(*$b_{j,n-1} \neq 0$ *) for some* j *then* ω *is not equivalent to* $\omega(0)$.

The problem of classification of forms with polynomial (in divided powers) coefficients has been solved completely only in the case of three variables [13]. In addition, the dimensions of all simple non-alternating Hamiltonian Lie algebras are found.

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Nullstellensatz theorem for skew PBW extensions

Oswaldo Lezama

Abstract. In this talk we define the algebraic sets and the ideal of points for bijective skew PBW extensions with coefficients in left Noetherian domains. Some properties of affine algebraic sets of commutative algebraic geometry will be extended, in particular, a Zariski topology will be constructed. Assuming additionally that the extension is quasi-commutative with polynomial center and the ring of coefficients is an algebraically closed field, we will prove an adapted version of Hilbert's Nullstellensatz theorem that covers the classical one. The Gröbner bases of skew PBW extensions will be used for defining the algebraic sets and for proving the main theorem. Many key algebras and rings coming from mathematical physics and non-commutative algebraic geometry are skew PBW extensions (see **Fajardo, W., Gallego, C., Lezama, O., Reyes, A., Suárez, H., and Venegas, H.**, *Skew PBW Extensions: Ring and module theoretic properties, matrix and Gröbner methods, applications*, Algebra and Applications 28, Springer, 2020).

Key words and phrases. Non-commutative algebraic geometry, algebraic sets, ideals of points, Hilbert's Nullstellensatz theorem, non-commutative Gröbner bases, skew PBW extensions.

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On the least primitive root of $2p + 1$

Makeshwari M

A joint work with V. P. Ramesh and Saswati Sinha

Abstract. The generators of the multiplicative group of integers modulo n, $(\mathbb{Z}/n\mathbb{Z})^*$ are called *primitive roots of* n. Gauss proved that $(\mathbb{Z}/n\mathbb{Z})^*$ *is cyclic if and only if* $n = 2, 4, p^k$ *or* $2p^k$ *for any odd prime* p *and any natural number* k [1]. Furthermore, Gauss conjectured that 10 *is a primitive root for infinitely many primes* p [1]. Later Artin conjectured that *for any* $a \neq \pm 1$ *and not a perfect square, there exist infinitely many primes* p *for which* a *is a primitive root.* A prime p is said to be a *Sophie Germain prime* if $2p + 1$ is also a prime. And, it is a conjecture that there is an infinitude of such primes. In this presentation, we present a necessary and sufficient condition for 2 to be a primitive root of $2p + 1$ and also a few more observations on $2p + 1$ [2].

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Conjugacy Preserving and Normal Automorphism Groups of the Generalized Dihedral Groups

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Abstract. Let H be a finite abelian group and $Dih(H)$ denote the generalized dihedral group on H. We use the notations $A = Aut(Dih(H))$ to denote the automorphism group of $Dih(H)$. If $x \in Dih(H)$, then the conjugacy class of $Dih(H)$ with representative x is denoted by $x^{Dih(H)}$. Suppose $Aut(Dih(H))$ denotes the full automorphism group of $Dih(H)$. Set $Aut_n(Dih(H)) = \{\alpha \in Aut(Dih(H)) \mid \alpha(N) = N;\ \forall N\ s.t.\ N \trianglelefteq Dih(H)\}$ and $Aut_{cc}(Dih(H)) = \{\alpha \in A \mid \alpha(x) \in x^{Dih(H)};\ \forall\ x\ \in Dih(H)\}$. In this paper, the group structure of $Aut_{cc}(Dih(H))$ and $Aut_n(Dih(H))$ are reported.

Introduction

Let H be an abelian group. The generalized dihedral group $Dih(H)$ is a the semidirect product of H by the cyclic group Z_2 in which Z_2 is acting on H by inverting elements. The set of all involutions of H together with the identity elements is the center of $Dih(H)$.

An automorphism α of a group G is said to be *conjugacy class preserving* if for every element $x \in G$, x and $\alpha(x)$ are conjugate in G. The set of all such automorphisms is denoted by $Aut_{cc}(G)$ and called the *conjugacy class preserving automorphism group* of G. If $Aut_i(G)$ denotes the inner automorphism group of G then it is easy to see that $Aut_i(G) \trianglelefteq Aut_c(G) \trianglelefteq Aut(G)$. The first appearance of the notion of conjugacy preserving automorphism is Note B of the famous book of Burnside [1] in which he asserted that $\frac{Aut_c(G)}{Aut_i(G)}$ is abelian.

An automorphism of a finite group G is said to be normal, if it takes each normal subgroup of G to itself. The set of all such automorphisms is denoted by $Aut_n(G)$ and it is easy to see that $Aut_{cc}(G) \trianglelefteq Aut_n(G) \trianglelefteq Aut(G)$ [2].

An elementary abelian group of order p^n, p is prime, is denoted by $E(p^n)$. Suppose A is a finite abelian group of even order. Define $E(A)$ to be the set of all involutions together with the identity element of A. It is easy to see that $E(A)$ is the largest elementary abelian 2-subgroup of A. If G has odd order, then by definition $E(A) = \{e\}$. The conjugacy class of a given group G with the representative x is denoted by x^G. The notations G' and $Z(G)$ are used for the center and derived subgroup of G. The semi-direct product of a group H by a group K given by the homomorphism $\alpha : K \longrightarrow Aut(K)$ is denoted by $H \times_\alpha K$.

We refer to the famous book of Robinson [3] for all concepts and symbols not presented here. Our results are checked by the computer algebra system GAP [4].

Conclusion

It is clear from definition that $Dih(A)$ is abelian if and only if A is an elementary abelian $2-$group. Since conjugacy classes of abelian groups can be obviously obtained, we will consider the case that A is not an elementary abelian 2-group. To calculate the conjugacy class preserving automorphism group of $Dih(H)$, we have to first obtain its conjugacy classes.

Theorem 1. *The conjugacy classes of* $Dih(A)$ *are* $\{x\}, x \in E(A)$; $x^{Dih(A)} = \{x, x^{-1}\}$, $x \in A \setminus E(A)$; *and* $(bh)^{Dih(A)} = (bh^{-1})A^2$, $h \in A$. *In particular, the generalized dihedral group* $Dih(A)$ *has exactly* $\frac{1}{2}(|A| + 3|E(A)|)$ *conjugacy classes.*

We are now ready to compute the normal subgroups of $Dih(H)$ in general. We note that $Aut(Dih(H)) = Hol(H)$, where $Hol(H)$ denotes the holomorph of the abelian group H. The generalized dihedral group has two types of subgroups. The first one is the subgroups of H, and the second type are subgroups with an element from $Dih(H) \setminus H$. It has to be checked the normality condition in subgroups of each type. It is easy to see that all subgroups of H is normal in $Dih(H)$ and so we have to check the subgroups of the second form. We have:

Theorem 2. *Let* A *be an abelian group not isomorphic to an elementary abelian 2-group. Then every normal subgroup of* $Dih(A)$ *is a subgroup of* A *or it has the form* $L \cup b(xL)$ *such that* L *is a subgroup of* A *containing* A^2. *Moreover, if* L *is a subgroup of* A *containing* A^2, *then for every element* $y \in A$, $L \cup byL$ *is a normal subgroup of* $Dih(A)$.

To compute the conjugacy class preserving automorphism group $Aut_{cc}(Dih(H))$, we apply the information provided by Lemma 1. Our first main result is as follows:

Theorem 3. $Aut_{cc}(Dih(A)) \cong Dih(A^2)$.

To calculate the normal automorphism group $Aut_n(Dih(H))$, we first calculate the normal automorphism group of H.

Theorem 4. *Let* H *be an abelian group and* A *be a cyclic subgroup* H *with maximum possible order. Then* $Aut_n(H) \cong Aut(A)$.

We are now ready to state our second main result as follows:

Theorem 5. $Aut_n(Dih(H)) = H^2 \times_\rho Aut_n(H)$, *where* $\rho : Aut_n(H) \longrightarrow Aut(H^2)$ *is given by* $\rho(f) = f|_{H^2}$.

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Twin H_4 in quaternions, amalgamations and Iwahori-Matsumoto decompositions

Jun Morita

Abstract. We find two root systems of type H_4 in quaternions. They create some infinite root system Σ. We will study it using amalgamations and Iwahori-Matsumoto decompositions.

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Rationally isotropic quadratic spaces are locally isotropic

Ivan Panin

Abstract. A conjecture due to Colliot-The'l'ene asserts that for a regular local ring R with its fraction field K each R-quadratic space (V,q), which is isotropic over K is isotropic over R (it is supposed that 2 is a unit in R). That is there exists a unomordular vector v in V with q(v)=0. This conjecture is proved for regular local rings R containing a field due to results by the speaker (2009), joint results of K.Pimenov and the speaker (2010), and a result by S.Scully (2018). The case of mixed characteristic ring R is widely open. In the talk a resent result by the speaker on the mixed characteristic case will be presented. We solve the conjecture in positive for a rather wide class of mixed characteristic regular local rings R. Our approach is based on new geometric presentation lemmas.

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POWERS IN FINITE ORTHOGONAL AND SYMPLECTIC GROUPS: A GENERATING FUNCTION APPROACH

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Abstract. For an integer $M \geq 2$ and a finite group G, an element $\alpha \in G$ is called an M-th power if it satisfies $A^M = \alpha$ for some $A \in G$. In this article, we will deal with the case when G is finite symplectic or orthogonal group over a field of order q. We introduce the notion of M^*-power SRIM polynomials. This, amalgamated with the concept of M-power polynomial, we provide the complete classification of the conjugacy classes of regular semisimple, semisimple, cyclic and regular elements in G, which are M-th powers, when $(M, q) = 1$. The approach here is of generating functions, as worked on by Jason Fulman, Peter M. Neumann, and Cheryl Praeger in the memoir "A generating function approach to the enumeration of matrices in classical groups over finite fields". As a byproduct, we obtain the corresponding probabilities, in terms of generating functions.

1. Introduction

1.1. **Question in the general context.** The motivation behind this work dates back to the work of two of the great mathematicians of the last century, A. Borel and E. Waring. Given an element $w \in \mathcal{F}_l$ (the free group on l generators), the map associated with w by plugging elements of G^l in w, is called a *word map*. It was proved by A. Borel (in [1]) (and later by Larsen independently in [11]) that given a semisimple algebraic group $\mathcal{G}$ and a word map $w : \mathcal{G}^l \longrightarrow \mathcal{G}$, it is a dominant map. The image of w will be denoted as $w(\mathcal{G})$ hereafter. The result due to Borel reveals the surprising result that $w(\mathcal{G})^2 = \mathcal{G}$. On the other hand, it was E. Waring, who mentioned in his paper "Meditationes Algebraicae" that, "every natural number is a sum of at most 9 cubes; every natural number is a sum of at most 19 fourth powers; and so on". These two problems gave rise to the following general question in the context of group theory.

Question 1.1. Given a group G and a word w on l generators, does there exists a $m(w) \in \mathbb{N}$ such that $(w(G))^{m(w)} = G$?

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This question is known as *Waring type problem* in group theory and has attracted an ample amount of attention from mathematicians in the past half-century. Substantial progress has been made and many fundamental questions are solved, using a wide spectrum of tools, including representation theory, probability, and geometry. A recent breakthrough in this direction is the affirmation of Ore's conjecture (which states that the commutator map corresponding to the word $xyx^{-1}y^{-1} \in F_2$ is surjective in the case of finite non-abelian simple groups), by Liebeck, O'Brien, Shalev and Tiep [13], using the methods from character theory. They proved that if G is any quasisimple classical group over a finite field, then every element of G is a commutator, using character-theoretic results due to Frobenius. In [12], the results about the product of squares in the finite non-abelian simple groups are proved. It was proved that every element of a non-abelian finite simple group G is a product of two squares. For a survey of these results and further problems in the context of group theory, we refer the reader to the excellent survey article due to Shalev [15].

In this article, we will be interested in the map associated with $w = x^M \in \mathcal{F}_1$ where $M \geq 2$ is an integer. This is a part of the ongoing project, where we intend to draw a conclusion about the image size of the power maps for finite groups of Lie type. The complete solution to this for the case of $\mathrm{GL}(n, q)$ has been described in [10]. But the existence of a root in the general linear group does not guarantee the existence of a root in the symplectic or orthogonal group. This has been a great motivation behind this work. The asymptotics of the powers in finite reductive groups has been pursued in [9]. Indeed, the authors therein estimate the proportion of regular semisimple, semisimple and regular which are M-th powers in the concerned groups, as q tends to infinity.

We will be giving the exact ratio for the symplectic and orthogonal groups over a finite field $\mathbb{F}_q$. We restrict ourselves to the case $(M, q) = 1$, as the case $(M, q) \neq 1$ is more intricate and will be followed up in future work. Our main results are Theorem **??** (and Theorem **??**), Theorem **??** (and Theorem **??**), Theorem **??** (and Theorem **??**, Theorem **??**) concerning generating function for the probability of a separable, semisimple, cyclic, and regular element respectively to be an M-th power in symplectic groups (and orthogonal groups). In follow-up work, we will be also looking into the case of power maps in exceptional groups of Lie type.

1.2. **Methodology.** We take the methods of statistical group theory, where generating functions play a key role. Before describing this, note that if $x \in G$ is an M-th power, then so are all conjugates of x. Hence to solve the question, it is admirable that we first find the conjugacy classes, which are M-th powers. The conjugacy class of finite orthogonal and symplectic groups is given by the combinatorial data consisting of self-reciprocal monic polynomials and signed symplectic or orthogonal partitions. The first description of conjugacy classes in these groups was discussed in the paper of Wall (see

[16]). The enumeration for the conjugacy classes is done with the machinery of generating functions along with the results of G. Wall.

Coming back to the viewpoint of statistical group theory, another way of looking into conjugacy classes is via cycle indices, which was introduced by Pólya for the symmetric group in the paper [14]. This can be briefly described as follows. For $\pi \in S_n$, let $a_i(\pi)$ denotes the number of i-cycles in π. Recall that in S_n, the number of elements with a_i many i-cycles is given by $n!/\prod_{i=1}^{n} a_i! i^{a_i}$. This along with the Taylor expansion of e^z gives that

$$\sum_{n=0}^{\infty} \frac{u^n}{n!} \sum_{\pi \in S_n} \prod_i x_i^{a_i(\pi)} = \prod_{m=1}^{\infty} e^{x_m u^m / m}.$$

Since then the concept of cycle index has been developed for various groups and has been used to derive exemplary results. For example in [5] cycle indices for the finite classical groups more precisely for unitary, symplectic, and orthogonal groups are studied. In the symplectic and in the orthogonal case it is assumed that q is odd. For the orthogonal groups a mixed cycle index is defined, taking into account both groups $\mathrm{O}^+(n,q)$ and $\mathrm{O}^-(n,q)$ at the same time. In the memoir [6], the authors consider probabilistic properties for classical groups over a fixed finite field of cardinality q when the rank goes to infinity. The results are about the asymptotics of corresponding probability. These results are very important and already have been used in many contexts, including the design of algorithms in group theory, random generation of simple groups, monodromy groups of curves, and derangements. Similar works have been pursued in an enormous amount of texts and a philomath is suggested to look into [2], [3], [4], [5], [8], [7] to have better understanding of this direction. We will be using these cycle indices for finite orthogonal and symplectic groups, drawn from [8] and some special polynomials to conclude our result. For reasons coming from the theory of cycle indices, the results for orthogonal groups rely on the results about symplectic groups.

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Fields of U-invariants

A.N. Panov

Abstract. The general linear group $\mathrm{GL}(n)$ acts on the linear space of matrix m-tuples by the adjoint action. The action of $\mathrm{GL}(n)$ induces the action of the unitriangular subgroup U. We present the system of free generators of the field of U-invariants.

The general linear group $\mathrm{GL}(n)$ over a field K acts on the space $\mathcal{H} = \mathrm{Mat}(n)\oplus\ldots\oplus\mathrm{Mat}(n)$ of matrix m-tuples by the adjoint action $\mathrm{Ad}_g(X_1,\ldots,X_m) = (gX_1g^{-1},\ldots,gX_mg^{-1})$. The action of the group $\mathrm{GL}(n)$ on $\mathcal{H}$ defines the representation

$$\rho(g)f(X_1,\ldots,X_m) = f(g^{-1}X_1g,\ldots,g^{-1}X_mg)$$

of the group $\mathrm{GL}(n)$ in the space of regular functions $K[\mathcal{H}]$. This representation is extended to the action of $\mathrm{GL}(n)$ on the field of rational functions $K(\mathcal{H})$. For a given subgroup $G \subseteq GL(n)$, the problem is to describe the algebra (respectively, the field) of invariants with respect to the action of G on $\mathcal{H}$.

In the case $G = \mathrm{GL}(n)$ (or $G = \mathrm{SL}(n)$), this problem is solved in the framework of the classical theory of invariants in tensors (see [1, 2, 3]). The algebra of $\mathrm{GL}(n)$-invariants is generated by the system of polynomials $\mathrm{Tr}(A_{i_1}\cdots A_{i_p})$, where $1 \leq i_1,\ldots,i_p \leq m$.

The group $\mathrm{GL}(n)$ contains the subgroup of unitriangular matrices $U = \mathrm{UT}(n)$, which consists of the upper triangular matrices with ones on the diagonal. Since U acts on $\mathcal{H}$ by unipotent transformations, the field $K(\mathcal{H})^U$ is a pure transcendental extension of K [4]. Our goal is to present a system of free generators of $K(\mathcal{H})^U$.

For $m = 2$, we have $\mathcal{H}_2 = \mathrm{Mat}(n)\oplus\mathrm{Mat}(n)$. Let $\{x_{ij}\}_{i,j=1}^n$ and $\{y_{ij}\}_{i,j=1}^n$ be the systems of standard coordinate functions on the first and second components of $\mathcal{H}_2$. Consider two matrices

$$X = (x_{ij})_{i,j=1}^n \qquad \text{and} \qquad Y = (y_{ij})_{i,j=1}^n.$$

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For two positive integers a and b, we denote by $[a,b]$ the subset of integers $a \leq i \leq b$. For any integer $1 \leq i \leq n$, let i' be the symmetric number to i with respect to the center of the segment $[1,n]$. We have $i' = n - i + 1$.

For the pair $i' \leq j$ (i.e. (i,j) lies on or below the anti-diagonal), let $M_{ij}(X)$ be the minor of order i' of the matrix X with the system of rows $[i,n]$ and columns $[1, i'-1] \sqcup \{j\}$.

For the pair $j \leq k$, let $N_{jk}(Y)$ be the minor of order k' of the matrix Y with the system of rows $\{j\} \sqcup [k+1, n]$ and columns $[1, k']$.

Let $i' < k$. This is equivalent to the pair (i,k) lies below the ant-diagonal. We define the polynomial

$$P_{ik}(X,Y) = \sum_{i' \leq j \leq k} M_{ij}(X) N_{jk}(Y). \qquad (1)$$

Proposition. The polynomials $\{P_{ik}(X,Y): \ i' < k\}$, and $D_k(X)$, $D_k(Y)$, where $1 \leq k \leq n$, are U-invariant.

We denote

$$P_{ik}(X) = P_{ik}(X,X) = \sum_{i' \leq j \leq k} M_{ij}(X) N_{jk}(X). \qquad (2)$$

Corollary. The polynomials $\{P_{ik}(X): \ i' < k\}$ are U-invariant.

For each $1 \leq i \leq n$, let $D_k(X)$ stand for the lower left corner minor of order k' of the matrix X. Observe that $D_k(X) = M_{k,k'}(X) = N_{k,k}(X)$.

In the case $m = 1$, we have $\mathcal{H} = \mathrm{Mat}(n)$. The group U acts on $\mathrm{Mat}(n)$ by the adjoint representation.

Theorem 1. The field $K(\mathrm{Mat}(n))^U$ is freely generated over K by the system of polynomials

$$\{P_{ik}(X): \ i' < k\} \sqcup \{D_k(X): \ 1 \leq k \leq n\}.$$

This system of free generators of $K(\mathrm{Mat}(n))^U$ is not unique. For the other approach see [5, 6].

For an arbitrary m and the linear space $\mathcal{H} = \mathrm{Mat}(n) \oplus \ldots \oplus \mathrm{Mat}(n)$ of matrix m-tuples, we define the following systems of polynomials:

$$\mathbb{P}_{1,\ell} = \{P_{ik}(X_1, X_\ell): \ \ 1 \leq i' < k \leq n\} \ \text{ for each } \ 2 \leq \ell \leq m,$$

$$\mathbb{P}_{\ell} = \{P_{ik}(X_\ell): \ \ 1 \leq i' < k \leq n\} \ \text{ for each } \ 1 \leq \ell \leq m,$$

$$\mathbb{D}_{\ell} = \{D_k(X_\ell): 1 \leq k \leq n\} \ \text{ for each } \ 1 \leq \ell \leq m.$$

Theorem 2. The field $K(\mathcal{H})^U$ is freely generated over K by the system of polynomials

$$\left(\bigcup_{\ell=2}^{m} \mathbb{P}_{1,\ell}\right) \cup \left(\bigcup_{\ell=1}^{m} \mathbb{P}_{\ell}\right) \cup \left(\bigcup_{\ell=1}^{m} \mathbb{D}_{\ell}\right).$$

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Automorphism groups of rigid affine surfaces: the identity component

Alexander Perepechko

Let Y be an affine variety over an algebraically closed field $\mathbb{K}$ of characteristic zero. We say that Y is *rigid* if it admits no effective action of the additive group of the field $\mathbb{G}_a$.

Conjecture (P.–Zaidenberg'13). *If Y is a rigid affine algebraic variety over $\mathbb{K}$, then the identity component of the automorphism group* $\mathrm{Aut}^\circ(Y)$ *is an algebraic torus of rank* $\leq \dim Y$.

We prove this conjecture in the case of a normal affine surface Y. We work with minimal completions (X, D) of Y with a normal crossing boundary divisor D and use the theory of birational transformations of the dual graph of the boundary divisor.

A *extremal linear segment* S of the dual weighted graph $\Gamma(D)$ is a maximal linear subgraph, which is either connected to $\Gamma(D) \setminus S$ by an edge from its end vertex or not connected at all. It is called *non-admissible*, if it contains a vertex of non-negative weight, that is, the corresponding curve in D has a non-negative self-intersection index. We present the following theorem.

Theorem (P.–Zaidenberg [1]**).** *If the surface Y is rigid, then all completions (X, D) have no non-admissible extremal linear segments in $\Gamma(D)$, and the identity component* $\mathrm{Aut}^\circ(Y)$ *is an algebraic torus of dimension* ≤ 2.

Otherwise, any completion (X, D) has a non-admissible extremal linear segment in $\Gamma(D)$, and $\mathrm{Aut}^\circ(Y)$ *contains an infinite-dimensional abelian unipotent subgroup.*

We also present a complete list of birational classes of dual graphs, which have a unique minimal model. All other dual graphs admit an infinite number of minimal models. The talk is based on the joint work with Mikhail Zaidenberg [1].

affine surface, automorphism group, completion, normal crossing divisor

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Orbits of Algebraic Groups and Classification Problems

Vladimir L. Popov

Abstract. There are many examples where the problem of classifying algebraic objects of a certain type is reformulated as that of classifying orbits of some algebraic group action. The talk is aimed to discuss the decidability of the equivalence problem for two objects of the considered type in such cases.

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Matrix divisors and Chevalley groups

O.K.Sheinman

Abstract. Matrix divisor is the historically first equivalent of the notion of holomorphic G-bundle on a Riemann surface. Matrix divisors have been introduced by A.Weil (1938), and later considered by A.Grothendieck in his talk at the Bourbaki seminar (1957), and by A.N.Tyurin (1964-65).

Matrix divisors can be defined as 0-cochains on the Riemann surface with coefficients in a Chevalley group over the field of rational functions. Based on this definition, we propose canonical form of a matrix divisor and find out moduli of matrix divisors.

By Riemann-Roch theorem, in the case G is simple, the dimension of the moduli space of stable G-bundles is equal to $(\dim G)(g-1)$ where g is genus of the Riemann surface. For the classical series of root systems, and for G_2, we present the moduli space of matrix divisors of the same dimension and conjecture that these are exactly the divisors of stable bundles.

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Matrix Waring Problem

Anupam Singh

The classical Waring problem in number theory asks if, for a given positive integer k, there exists $g(k)$, such that every natural number is a sum of $g(k)$ many k-th powers, and $g(k)$ is smallest with this property. That is, the function $X_1^k + X_2^k + \cdots + X_{g(k)}^k$ represents all natural numbers. This problem has a long history starting from Waring in 1770. Waring problem has also been considered for various other rings, such as the ring of integers, polynomial rings, etc. Modern versions consider the analogues question over objects with noncommutative structures, such as groups, Lie algebras, matrix algebras, etc. Various generalisation of this problem considers more general words, instead of just power words, and at times ignore small size for better bounds. For example, Shalev [6] showed that for every finite (nonabelian) simple group of sufficiently high order every element can be expressed as values of word w of length 3. This was later improved to 2 by Larsen, Shalev and Tiep [3].

Larsen conjectured that a similar result should hold for matrices over finite fields. In other words, if R denotes a commutative ring with unity, then the Matrix Waring Problem would be to address whether matrices over R, possibly when the entries are "large" enough, can be expressed as a sum of two kth powers (of matrices). The goal of this article is to answer this question in the case where R is a finite field $\mathbb{F}_q$, with q sufficiently large. We have, for all integers $k \geq 1$, there exists a constant C_k depending only on k, such that for all $q > C_k$ and for all $k \geq 1$ every matrix in $M_n(\mathbb{F}_q)$ is a sum of two k-th powers.

This work is done in collaboration with Krishna Kishore. In [1], he proved that it can be written as a sum of at most three k-th powers which we have improved to 2 now. The key idea is to use powers in $GL(n, q)$ from [2], and Lang-Weil's results on the number of solutions to equations over finite fields [4, 5].

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Rook placements in G_2 and F_4 and associated coadjoint orbits

Matvey Surkov

Abstract. Let N be a nilpotent complex Lie group, $\mathfrak{n}$ be its Lie algebra. Given a rook placement D and a map from this rook placement into $\mathbb{C}^{\times}$, one can define the linear form on $\mathfrak{n}$. We are interested in the coadjoint orbits of such forms. Let ξ_1 and ξ_2 be distinct maps from D to $\mathbb{C}^{\times}$, let Ω_{D,ξ_1}, Ω_{D,ξ_2} be the correspondent orbits. M. Ignatev give a conjecture that $\Omega_{D,\xi_1} \neq \Omega_{D,\xi_2}$. I will prove this conjecture for the cases $\Phi = G_2$ and F_4.

Let N be a nilpotent complex Lie group, $\mathfrak{n}$ be its Lie algebra, and $\mathfrak{n}^*$ be its dual space. The group N acts on the Lie algebra $\mathfrak{n}$ via an adjoint representation; the dual representation in the space $\mathfrak{n}^*$ is called coadjoint. According to the orbit method of A.A. Kirillov, the orbits of the coadjoint representation play a key role in the representation theory of the group N [6].

We are interested in the case when N is a unipotent radical of a Borel subgroup B of a simple complex algebraic group G. The complete classification of coadjoint orbits in $\mathfrak{n}^*$ is a wild problem. Therefore, of particular interest is the study of certain important classes or series of orbits. Almost all the orbits that have been studied to date are so-called orbits associated with rook placements in root systems.

Let Φ be the root system of the group G, Φ^+ be the set of positive roots with respect to B, the set $\{e_\alpha, \alpha \in \Phi^+\}$ be the basis of the algebra $\mathfrak{n}$ consisting of root vectors, the set $\{e^*_\alpha, \alpha \in \Phi^+\}$ be the dual basis of space $\mathfrak{n}^*$. A rook placement is a subset $D \subset \Phi^+$ consisting of roots with pairwise non-positive scalar products. For any mapping $\xi\colon D \to \mathbb{C}^{\times}$, we define the linear form $f_{D,\xi} = \sum\limits_{\beta \in D} \xi(\beta) e^*_\beta \in \mathfrak{n}^*$; let $\Omega_{D,\xi} \subset \mathfrak{n}^*$ be its orbit. We will say that the orbit $\Omega_{D,\xi}$ is associated with the rook placement D. For example, in the case $\Phi = A_{n-1}$, all orbits of maximal dimension are associated with the same orthogonal rook placement, called the Kostant cascade.

We call a rook placement D non-singular if the fact that α and β are in D implies that $\alpha - \beta \notin D$. For example, for $\Phi = A_{n-1}$ all rook placements are non-singular.

Orbits associated with orthogonal rook placements (in particular, with Kostant cascades) were studied in detail in [2], [3], [7], [8].

During the study, the following conjecture arose.

Conjecture. *Let D be a non-singular orthogonal rook placement, ξ_1, ξ_2 be different maps from D to $\mathbb{C}^\times$. Then Ω_{D,ξ_1} and Ω_{D,ξ_2} do not coincide.*

For $\Phi = A_{n-1}$ this follows from the results of A.N. Panov [8]. For the remaining classical series B_n, C_n, D_n, the proof of the conjecture essentially reduces to the case A_{n-1}. In [5] M.V. Ignatiev and A.A. Shevchenko proved that the conjecture is true for $\Phi = E_6, E_7$ or E_8 for some (but not all) rook placement. The main result of my report is as follows.

Theorem. *Conjecture is true in cases $\Phi = G_2$ and F_4*

For G_2, this is an easy exercise, while for F_4, a modification of the argument from [6] is required, which, in fact, also reduces the problem to the case A_{n-1}, where stronger results of C. Andre [1] can be applied.

The talk is based on joint work with M.V. Ignatev [4].

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Finite and infinite groups generated by involutions

Anatoly Vershik

We consider a wide class of groups, which generalizes Coxeter groups. Namely, we study the groups of symmetries of poset numerations, in particular, the groups of symmetries of the set of Young tableaux with given Young diagram with n boxes. The group is generated by $n-1$ involutions of the replacements $(i, i+1)$, $i = 1, \ldots, n$. The symmetric group corresponds to the simplest diagram $(n, 1)$. The answer for an arbitrary diagram is not known. So far among the obtained groups there only appear symmetric groups, alternating groups and some Coxeter groups (results of the speaker, N.V.Tsilevich, and the graduate students M.Germanskov and P.Pozdeev). The most intriguing questions are related to (infinite) groups which correspond to infinite diagrams, to their structure and representations. This work is carried out as part of a project supported by an RNF grant 21-11-00152

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Groups with commutator relations of type A

Egor Voronetsky

Abstract. The general linear group $\mathrm{GL}(n, A)$ over an associative unital ring A contains elementary transvections $t_\alpha(p)$ for α from the root system of type A_{n-1} and $p \in A$. These elements satisfy $t_\alpha(p+q) = t_\alpha(p)\, t_\alpha(q)$ and the Chevalley commutator formula. We show a partial converse to this fact: suppose that a group G has a family of subgroups parametrized by the root system of type A_{n-1} and these subgroups satisfy the Chevalley commutator formula. Then under an additional natural assumption there is a unique homomorphism from the Steinberg group of a generalized matrix ring to G mapping the root subgroups to the distinguished subgroups of G.

Let A be a unital associative ring and $n \geq 4$. The group $\mathrm{GL}(n, A)$ contains the elementary transvections $t_{ij}(p) = 1 + pe_{ij}$ for $i \neq j$ and $p \in A$ satisfying the Steinberg relations

$$\begin{aligned} t_{ij}(p+q) &= t_{ij}(p)\, t_{ij}(q); \\ [t_{ij}(p), t_{jk}(q)] &= t_{ik}(pq) \text{ for } i \neq k; \\ [t_{ij}(p), t_{kl}(q)] &= 1 \text{ for } j \neq k \text{ and } i \neq l. \end{aligned}$$

The unstable Steinberg group $\mathrm{St}(n, A)$ is generated by the elements $x_{ij}(p)$ for $1 \leq i \neq j \leq n$ and $p \in A$ satisfying the Steinberg relations, so there is a canonical homomorphism

$$\mathrm{St}(n, A) \to \mathrm{GL}(n, A), x_{ij}(p) \mapsto t_{ij}(p).$$

Now suppose that we have a group G with subgroups G_{ij} for $1 \leq i < j \leq n$ such that

$$\begin{aligned} [G_{ij}, G_{jk}] &\leq G_{ik} \text{ for } i \neq k; \\ [G_{ij}, G_{kl}] &= 1 \text{ for } j \neq k \text{ and } i \neq l. \end{aligned}$$

In other words, G has A_{n-1}-commutator relations in the sense of [2, chapter I]. If G in addition has Weyl elements n_{ij} permuting the subgroups G_{ij}, then under natural assumption there are a unital associative ring A and a homomorphism $f \colon \mathrm{St}(n, A) \to G$ such that f maps $A \cong x_{ij}(A)$ isomorphically to G_{ij}, see [3].

We are interested in the more general case where G does not contain Weyl elements. For example, let R be a generalized matrix ring, i.e. a non-unital associative ring with a Peirce decomposition

$$\begin{aligned} R &= \bigoplus_{1\leq i,j\leq n} R_{ij}; \\ R_{ij}R_{jk} &\leq R_{ik}; \\ R_{ij}R_{kl} &= 0 \text{ for } j \neq k. \end{aligned}$$

An element $x \in R$ is called quasi-invertible if it is invertible with respect to the monoid operation $x \circ y = xy + x + y$. Let $G = R^\circ$ be the group of quasi-invertible elements of R. It contains the subgroups $G_{ij} = R_{ij}$ satisfying the commutator relations. Clearly, there are no Weyl elements in G in general, e.g. for

$$\begin{aligned} R &= \mathrm{M}(5, \mathbb{Z}); \\ R_{ij} &= \mathbb{Z}e_{ij} \text{ for } i, j < 4; \\ R_{i4} &= \mathbb{Z}e_{i4} \oplus \mathbb{Z}e_{i5} \text{ for } i < 4; \\ R_{4j} &= \mathbb{Z}e_{4j} \oplus \mathbb{Z}e_{5j} \text{ for } j < 4; \\ R_{44} &= \mathbb{Z}e_{44} \oplus \mathbb{Z}e_{45} \oplus \mathbb{Z}e_{54} \oplus \mathbb{Z}e_{55} \end{aligned}$$

the subgroups G_{ij} are not isomorphic.

Our main result [1] is the following: let G be a group with subgroups G_{ij} satisfying the commutator conditions and an additional natural assumption (a stronger version of the condition $[G_{ij}, G_{jk}] = G_{ik}$ for $i \neq k$). Then there is a generalized matrix ring R and a homomorphism from its Steinberg group to G mapping the root subgroups isomorphically to G_{ij}, so $R_{ij} \cong G_{ij}$.

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Elements with almost simple spectrum in representations of simple algebraic groups

Alexandre Zalesski

Abstract. In this talk I intend to report on some recent results on the spectra of semisimple elements of algebraic groups in their irreducible representations. Let G be an algebraic group, $g \in G$ a semisimple element and ρ an irreducible representation. Then $\rho(g)$ is said to have simple spectrum if all its eigenvalues are of multiplicity 1, and almost simple spectrum if at most one of the eigenvalues is of greater than 1 multiplicity. Most results are obtained jointly with D. Testerman.

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THE MOST NON-ALGEBRAIC COMPLEX TORI - AN ALGEBRAIC CONSTRUCTION

YURI G. ZARHIN

This is a report on a joint work with Tatiana Bandman (Bar-Ilan).

Definition. Let $\mathbb{Q}$ be the field of rational numbers and $g \geq 2$ an integer. We say that a a degree $2g$ polynomial $f(x) \in \mathbb{Q}[x]$ *without real roots* is *very irreducible* if it enjoys the following property.

The polynomial $f(x)$ is irreducible over $\mathbb{Q}$ and its Galois group over $\mathbb{Q}$ (which is apriori transitive) is doubly transitive.

Examples. It follows from results of I. Schur that the *truncated exponent*

$$\exp_{2g}(x) = \sum_{j=1}^{2g} \frac{x^j}{j!}$$

is very irreducible for all g. It follows from results of E. Selmer, E. Nart and N. Vila that if g is *not* congruent to 1 modulo 3 then the polynomial $x^{2g} + x + 1$ is very irreducible.

One may easily check that if $f(x)$ is a degree $2g$ very irreducible polynomial then the quotient $K_f := \mathbb{Q}[x]/f(x)\mathbb{Q}[x]$ is a purely imaginary number field of degree $2g$ that does not contain proper subfields except $\mathbb{Q}$. This implies that the only roots of unity in K_f are 1 and -1.

For each very irreducible $f(x)$ we construct a *simple* g-dimensional complex torus T_f that enjoys the following properties.

- (i) The endomorphism algebra of T_f is isomorphic to K_f.
- (ii) The Picard number of T_f is 0 and therefore the algebraic dimension of T is 0. In particular, T_f is *not* an abelian variety.
- (iii) If $T_f^{\vee}$ is the dual of T_f then $\mathrm{Hom}(T_f, T_f^{\vee}) = \{0\}$.
- (iv) The automorphism group $\mathrm{Aut}(T)$ of the complex Lie group T is isomorphic to $\mathbb{Z}^{g-1} \times \{\pm 1\}$.

These properties suggest that one may view T_f as transcendental counterparts of abelian varieties of CM type. (Notice that properties (ii)-(iv) follow from property (i) combined with simplicity of T_f.)

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For any given $g \geq 2$ we construct explicitly infinitely many very irreducible polynomials $f_n(x)$ of degree $2g$ with mutually non-isomorphic numbers fields K_{f_n}. This implies that the corresponding g-dimensional simple complex tori T_{f_n} are mutually non-isogenous.

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